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# $L_p$ Markov–Bernstein Inequalities for Freud Weights

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Let $W(x) := \exp(-Q(x))$, $x \in \mathbb{R}$, where $Q(x)$ is even and continuous in $\mathbb{R}$, $Q(0)=0$ and $Q''$ is continuous in $(0, \infty)$ with $Q'(x)>0$ in $(0, \infty)$, and for some $A, B>1$,

$$A \leqslant (xQ'(x))'/Q'(x) \leqslant B, \qquad x \in (0, \infty).$$

For example, $Q(x) := |x|^\alpha$, $\alpha > 1$ satisfies these hypotheses. Let $a_n$ denote the $n$th Mhaskar–Rahmanov–Saff number for $Q$, and

$$\varphi_n(x) := \max\left\{1-\frac{|x|}{a_n}, n^{-2/3}\right\}, \qquad n \geqslant 1, \quad x \in \mathbb{R}.$$

Let $1 \leqslant p < \infty$. We prove that for $n \geqslant 1$ and polynomials $P$ of degree at most $n$,

$$\|(PW)'\varphi_n^{-1/2}\|_{L_p(\mathbb{R})} \leqslant C\frac{n}{a_n}\|PW\|_{L_p(\mathbb{R})}.$$

This extends to $L_p$ the recent $L_\infty$ result of the authors, in which the essential feature is the introduction of the factor $\varphi_n^{-1/2}$. We also consider the case $A \leqslant 1$. The proofs are necessarily different from previous methods of extending $L_\infty$ inequalities to $L_p$, and involve Carleson measures. © 1994 Academic Press, Inc.

## 1. INTRODUCTION AND RESULTS

Throughout $\mathcal{P}_n$ denotes the class of real polynomials of degree at most $n$, and $C, C_1, C_2, \ldots$, denote positive constants independent of $n$, $P \in \mathcal{P}_n$ and

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$x \in \mathbb{R}$. The same $C$ does not necessarily represent the same constant in different occurrences. We use $\sim$ in the following sense: If $\{b_n\}_{n=0}^{\infty}$ and $\{c_n\}_{n=0}^{\infty}$ are sequences of non-zero real numbers, we write

$$b_n \sim c_n,$$

if there exist $C_1, C_2 > 0$ such that

$$C_1 \leqslant b_n/c_n \leqslant C_2, \qquad n \geqslant 1.$$

Similar notation is used for functions and sequences of functions.

The classical $L_p$ Markov–Bernstein inequality for $[-1, 1]$ involves the factor

$$\psi_n(x) := \min\left\{n, \frac{1}{\sqrt{1-x^2}}\right\}$$

and for any $0 < p < \infty$, has the form

$$\|P'\|_{L_p[-1,1]} \leqslant Cn \, \|P\psi_n\|_{L_p[-1,1]}, \qquad P \in P_n, \quad n \geqslant 1. \tag{1.1}$$

The usefulness of such inequalities in approximation theory, discretisation problems, quadrature and interpolation is well known.

There are many ways to proceed from the $L_\infty$ version of (1.1) to the general $L_p, p > 0$ case. One of the most versatile is a technique adapted, in spirit, from the large sieve of number theory, and involves $L_p$ Christoffel functions: See [13, 2, 3, 11] for details of the method. That method, and all others known to the authors, make essential use of the fact that uniformly for $x \in (-1, 1)$ and $n \geqslant 1$,

$$\psi_n(x) \sim \psi_{2n}(x).$$

In this paper, we present a new method, involving Carleson measures, to prove $L_p$ Markov–Bernstein inequalities when this last relation fails. The specific context in which we outline the method is $L_p$ Markov–Bernstein inequalities for Freud weights.

Recall that if $W := e^{-Q}$, where $Q: \mathbb{R} \to \mathbb{R}$ is even and continuous in $\mathbb{R}$, and of smooth polynomial growth at infinity, then we call $W$ a Freud weight [17]. Associated with $Q$ is the *Mhaskar–Rahmanov–Saff* number $a_u$ [14, 15, 19] the positive root of the equation

$$u = \frac{2}{\pi} \int_0^1 a_u t Q'(a_u t) \, dt / \sqrt{1-t^2}, \qquad u > 0. \tag{1.2}$$

It exists, for example, when $xQ'(x)$ is increasing in $(0, \infty)$, with limits 0 and $\infty$ at 0 and $\infty$ respectively. Following is our main result:

THEOREM 1.1. *Let* $W := e^{-Q}$, *where* $Q\colon \mathbb{R} \to \mathbb{R}$ *is even and continuous in* $\mathbb{R}$, $Q(0) = 0$, *and* $Q''$ *is continuous in* $(0, \infty)$, $Q'(x)$ *is positive in* $(0, \infty)$, *and for some* $A, B > 1$,

$$A \leqslant (xQ'(x))'/Q'(x) \leqslant B, \qquad x \in (0, \infty). \tag{1.3}$$

*Let*

$$\varphi_n(x) := \max\left\{1 - \frac{|x|}{a_n}, n^{-2/3}\right\}, \qquad x \in \mathbb{R}, \quad n \geqslant 1. \tag{1.4}$$

*Let* $1 \leqslant p < \infty$. *Then there exists* $C > 0$ *such that for* $n \geqslant 1$ *and* $P \in \mathscr{P}_n$,

$$\|(PW)' \varphi_n^{-1/2}\|_{L_p(\mathbb{R})} \leqslant C \frac{n}{a_n} \|PW\|_{L_p(\mathbb{R})}. \tag{1.5}$$

*Remarks.* (a) Markov-Bernstein inequalities of the form

$$\|P'W\|_{L_p(\mathbb{R})} \leqslant C \frac{n}{a_n} \|PW\|_{L_p(\mathbb{R})}, \qquad P \in P_n, \quad n \geqslant 1, \tag{1.6}$$

have been widely studied and applied in the literature [1, 4, 5, 7–9, 17, 18], especially in relation to converse theorems of approximation. In fact, (1.6) is a simple consequence of (1.5), since $|Q'(x)| = O(n/a_n)$ for $|x| \leqslant 2a_n$ (see (4.2), (4.3) below). Even (1.6) is new for the full generality of weights $W$ considered here, as previously additional conditions were required when in (1.3), $1 < A < 2$.

However, the essential feature of the theorem is the insertion of the factor $\varphi_n^{-1/2}$, which is large near $a_n$. For $p = \infty$, the inequality (1.5) was established in [9], and played an important role in establishing bounds for the orthogonal polynomials for the weight $W^2 = e^{-2Q}$ [10]. We believe that the $p < \infty$ case will also have applications.

(b) Methods used to prove (1.6) for various weights in [4, 5, 8, 11] include boundedness of dilated de la Vallee–Poussin sums, replacement of the weight over a suitable interval by polynomials of degree $O(n)$, or a technique adapted from the large sieve of number theory. All attempts to adapt these to the present situation failed, because they would require the same inequality to hold for polynomials of degree $2n$ as for $n$ (modulo a constant). However, it is not true that

$$\varphi_n(x) \sim \varphi_{2n}(x), \qquad x \in \mathbb{R}, \quad n \geqslant 1,$$

so (1.5) provides a different inequality for polynomials of degree $\leqslant n$ as compared to polynomials of degree $\leqslant 2n$.

Our method is given in Section 2: We adapt the complex and potential theoric methods from [9] to obtain local estimates for $(PW)'(x)$ in terms of the average of $|P(t)|\, W(|t|)$ on a semicircle centred on $x$ and then integrate: To return to the real line, we use a result about Carleson measures.

(c) The restriction $p \geqslant 1$ is unfortunate but we have been unable to find a device to circumvent it. One extension that should be fairly immediate is to Orlicz-space type inequalities

$$\int_{-\infty}^{\infty} \psi(|(PW)'(x)\, \varphi_n^{-1/2}(x)|^p)\, dx$$

$$\leqslant C_1 \int_{-\infty}^{\infty} \psi\left(C_2 \frac{n}{a_n} |(PW)(x)|^p\right) dx, \qquad P \in \mathscr{P}_n.$$

Here $p \geqslant 1$ and $\psi$: $[0, \infty) \to [0, \infty)$ is a convex function with $\psi(0) = 0$. The only missing ingredient is an inequality of the form

$$\int \psi(|U|^p)\, d\sigma \leqslant C \int_{-\infty}^{\infty} \psi(|f(x)|^p)\, dx,$$

valid for functions $f \in L_p(\mathbb{R})$ with Poisson integrals $U(z)$ in the upper half plane, and for Carleson measures $\sigma$. Possibly interpolation could be used to provide this missing step.

(d) The inequality (1.5) is almost certainly not true if we replace $(PW)'$ by $P'W$. In [9], this was proved for $p = \infty$, by nothing that if $T_n(x) = x^n + \cdots \in \mathscr{P}_n$ is an $L_\infty$ extremal polynomial in the sense that

$$\|T_n W\|_{L_\infty(\mathbb{R})} = \min\{\|PW\|_{L_\infty(\mathbb{R})} : P(x) = x^n + \cdots \in \mathscr{P}_n\},$$

then at the largest point of equioscillation of $T_n W$, $\zeta_n$ say, we have

$$|T_n' W|(\zeta_n) = Q'(\zeta_n)\, \|T_n W\|_{L_\infty(\mathbb{R})} \sim \frac{n}{a_n} \|T_n W\|_{L_\infty(\mathbb{R})},$$

while

$$\zeta_n = a_n\left(1 + O\left(\left(\frac{\log n}{n}\right)^{2/3}\right)\right).$$

(e) The above result does not apply to $Q(x) := |x|^\alpha$, $\alpha \leqslant 1$, since for such $Q$, $A = B = \alpha \leqslant 1$.

By omitting an interval of length $2\eta a_n$ about 0, we can still prove an analogue of Theorem 1.1:

THEOREM 1.2. *Let* $W := e^{-Q}$ *be as in Theorem* 1.1, *except that we only require* $A, B > 0$ *in* (1.3). *Let* $\eta > 0$, *and* $p \geq 1$, *and* $\varphi_n$ *be defined by* (1.4). *There exists* $C > 0$ *such that for* $n \geq 1$ *and* $P \in \mathscr{P}_n$,

$$\|(PW)' \varphi_n^{-1/2}\|_{L_p(|x| \geq \eta a_n)} \leq C \frac{n}{a_a} \|PW\|_{L_p(\mathbb{R})}. \tag{1.7}$$

Note that $Q'(0)$ need not exist for the weights in Theorem 1.2, whereas for the weights in Theorem 1.1, we have $Q'(0) = 0$. As indicated by the $L_\infty$ inequalities in [9], the $L_p$ inequalities over $[-\varepsilon a_n, \varepsilon a_n]$ have a different form to that in (1.7), but we shall not dwell on this point here.

The paper is organised as follows: In Section 2, we prove Theorems 1.1 and 1.2, but leave several technical details to later sections. In Section 3, we estimate the Carleson norms of certain measures $\sigma_n$, thereby proving Lemma 2.4. In Section 4, we prove Lemma 2.1, which relates certain entire functions to the weight $W$. In Section 5, we fill in some missing details in Lemma 2.2, concerning certain analytic functions and the weight $W$. Finally, in Section 6, we estimate the derivative of a certain quantity, establishing the last technical detail used in Section 2.

## 2. THE PROOF OF THEOREMS 1.1 AND 1.2.

We break this into several steps:

*Step* 1.

We replace the weight $W$ locally by an analytic one.
Given $x \in \mathbb{R}$, set

$$H_x(z) := e^{-[Q(x) + Q'(x)(z-x)]}, \qquad z \in \mathbb{C}. \tag{2.1}$$

Since $H_x^{(j)}(x) = W^{(j)}(x), j = 0, 1$, we obtain by Cauchy's formula,

$$(PW)'(x) = (PH_x)'(x) = \frac{1}{2\pi i} \int_{|z-x| = \varepsilon} \frac{PH_x(z)}{(z-x)^2} dz,$$

for any polynomial $P$ and for any $\varepsilon > 0$. Assuming $P$ has real coefficients, we obtain that

$$|(PW)'(x)| \leq \frac{1}{\pi\varepsilon} \int_0^\pi |(PH_x)(x + \varepsilon e^{i\theta})| \, d\theta. \tag{2.2}$$

The choice of $\varepsilon$ is suggested by (1.4), (1.5). For $x \in \mathbb{R}$ and $n \geq 1$, set

$$\varepsilon := \varepsilon_n(x) := \frac{a_n}{n} \begin{cases} (1-|x|/a_n + n^{-2/3})^{-1/2}, & |x| \leqslant a_n \\ n^{1/3}, & |x| \geqslant a_n \end{cases} \tag{2.3}$$

Note that uniformly for $n \geq 1$ and $x \in \mathbb{R}$,

$$\varepsilon_n(x) \sim \frac{a_n}{n} \varphi_n^{-1/2}(x).$$

LEMMA 2.1. *Let $\varepsilon_n(x)$ be defined by* (2.3) *and assume the hypotheses of Theorem* 1.1. *Then there exists $C > 0$ such that*

$$|H_x(x+\varepsilon_n(x)\,e^{i\theta})| \leqslant CW(|x+\varepsilon_n(x)\,e^{i\theta}|), \tag{2.4}$$

*for all $n \geq 1$, $\theta \in [0, \pi]$ and for all $x \in J_n$, where*

$$J_n := \{x \in \mathbb{R}: |x| \leqslant a_n(1+n^{-2/3})\}. \tag{2.5}$$

If $W$ only satisfies the conditions of Theorem 1.2, (2.4) holds for the range $J_n \cap \{x: |x| \geqslant \eta a_n\}$, any fixed $0 < \eta < 1$.

*Proof.* See Section 4. ∎

Replacing $\varepsilon$ in (2.2) by $\varepsilon_n$ as defined in (2.3), applying Hölder's inequality, and then integrating over $J_n$, we obtain, by (2.4),

$$\int_{J_n} |(PW)'\,\varepsilon_n|^p\,dx \leqslant C \int_{J_n} \left\{ \int_0^\pi |P(x+\varepsilon_n(x)\,e^{i\theta})\,W(|x+\varepsilon_n(x)\,e^{i\theta}|)|^p\,d\theta \right\} dx, \tag{2.6}$$

where $p \geq 1$ and $\varepsilon_n = \varepsilon_n(x)$.

*Step* 2.

Our next step is to replace $W$ globally an analytic weight.

This construction is well known (cf. [12, 14, 15, 19]), but for the reader's convenience, we provide some details of proof of the following lemma in Section 5.

LEMMA 2.2. *Assume the conditions of Theorem* 1.2. *Then, given $n \geq 1$, there exists a function $G$, that satisfies as following*:

(a) *$G$ is analytic in $\bar{\mathbb{C}} \setminus [-a_n, a_n]$, with a simple zero at infinity, and satisfies*

$$G(\bar{z}) = \overline{G(z)}. \tag{2.7}$$

*Moreover $G$ has boundary values $G(x \pm i0)$ that are continuous on $(-a_n, a_n)\backslash\{0\}$ and that satisfy*

$$|G^n(x \pm i0)| = W(x), \qquad x \in [-a_n, a_n]\backslash\{0\}. \tag{2.8}$$

*Moreover,*

$$|G^n(x)| > W(x), \qquad |x| > a_n. \tag{2.9}$$

(b) *If $W$ satisfies the conditions of Theorem 1.1, then there exists $C > 0$, independent of $n$, such that*

$$W(|x + \varepsilon_n(x)\, e^{i\theta}|) \leqslant C\,|G^n(x + \varepsilon_n(x)\, e^{i\theta})|, \tag{2.10}$$

*for all $x \in J_n$ and $\theta \in [0, \pi]$. If $W$ only satisfies the conditions of Theorem 1.2, then given $0 < \eta < 1$, (2.10) still holds (with $C = C(\eta)$) for $x \in J_n \cap \{x\colon |x| \geqslant \eta a_n\}$.*

In the sequel we assume that $W$ satisfies the conditions of Theorem 1.1. Applying (2.10), we deduce from (2.6) that

$$\int_{J_n} |(PW)'(x)\, \varepsilon_n(x)|^p\, dx \leqslant C \int_{J_n} \int_0^\pi |(PG^n)(x + \varepsilon_n(x)\, e^{i\theta})|^p\, d\theta\, dx. \tag{2.11}$$

Next, let us introduce a positive measure $d\sigma_n$ on the upper half plane, that is defined by

$$\sigma_n(S) := \int_{J_n} \int_0^\pi \chi_S(x + \varepsilon_n(x)\, e^{i\theta})\, d\theta\, dx, \tag{2.12}$$

where $S$ is any Borel set in the upper half plane, and $\chi_S$ is its characteristic function. With this definition, we rewrite (2.11) as

$$\int_{J_n} |(PW)'(x)\, \varepsilon_n(x)|^p\, dx \leqslant C \int |PG^n|^p\, d\sigma_n. \tag{2.13}$$

*Step* 3.

The next step involves the notion of a *Carleson measure*.

This is a positive measure $d\sigma$ on the upper half plane, that satisfies for some $C > 0$,

$$\sigma(K_{x_0, h}) \leqslant Ch, \tag{2.14}$$

for any square $K_{x_0, h}$ of the form

$$K_{x_0, h} = [x_0 - \tfrac{1}{2} h, x_0 + \tfrac{1}{2} h] \times [0, h], \tag{2.15}$$

where $x_0 \in \mathbb{R}$, $h > 0$. Note that these squares have base on the real axis and lie in the upper half plane. The smallest constant $C$ in (2.14) is called the *Carleson norm* $N(\sigma)$ of $\sigma$.

We also recall that the Hardy space $H^p, 0 < p < \infty$, on the upper half plane consists of all functions $f$ analytic there and satisfying

$$\|f\|_{H^p}^p := \sup_{y>0} \int_{-\infty}^{\infty} |f(x+iy)|^p \, dx < \infty.$$

Any $f \in H^p$ has non-tangential boundary values $f(x)$, as $z \to x$ from the upper half plane, for a.e. $x \in \mathbb{R}$, and there holds

$$\|f\|_{H^p}^p := \int_{-\infty}^{\infty} |f(x)|^p \, dx.$$

The following result is due to L. Carleson. For the proof, see [6, Thm. 5.6, p. 33, and Thm. 3.9, p. 63]:

LEMMA 2.3. *Let $d\sigma$ be a Carleson measure and $0 < p < \infty$. Then there exists a constant $C_p > 0$, depending only on $p$, such that for any $f \in H^p$,*

$$\int |f|^p \, d\sigma \leqslant C_p \, N(\sigma) \int_{-\infty}^{\infty} |f(x)|^p \, dx. \tag{2.16}$$

It turns out that

LEMMA 2.4. *The measure $d\sigma_n$, as defined in (2.12), is a Carleson measure, and its norm $N(\sigma_n)$ is bounded from above by a constant independent of $n$.*

We prove Lemma 2.4 in Section 3. Now, let us return to (2.13). Since $G$ has a zero at infinity, and more precisely is $O(1/z)$ there, the same is true for $PG^n$, provided $P \in \mathscr{P}_{n-1}$. So $PG^n \in H^p$ for any $p > 1$. (If $p = 1$, we would need to take $P \in P_{n-2}$). Applying Lemma 2.3 and Lemma 2.4, we may replace (2.13) by

$$\int_{J_n} |(PW)'(x) \, \varepsilon_n(x)|^p \, dx \leqslant C \int_{-\infty}^{\infty} |PG^n|^p \, dx, \qquad P \in \mathscr{P}_{n-1}. \tag{2.17}$$

*Step* 4.

We replace $PG^n$ by $PW$ in (2.17).

Now we are almost done. Since $PG^n$ is analytic in $\bar{\mathbb{C}}\setminus[-a_n, a_n]$ and vanishes at infinity, a simple application of Cauchy's formula yields

$$(PG^n)(z)=\frac{1}{\pi}\int_{-a_n}^{a_n}\frac{\operatorname{Im}(PG^n)(t+i0)}{t-z}\,dt, \qquad z\notin[-a_n, a_n]. \tag{2.18}$$

This relation is classical, but in the context of orthogonal polynomials on $\mathbb{R}$, was first used by E. A. Rahmanov.

Let us define

$$G^n(x):=G^n(x+i0), \qquad x\in(-a_n, a_n).$$

From (2.18), we see that the restriction of $PG^n$ to $(-\infty, -a_n)\cup(a_n, \infty)$ is the Hilbert transform of the function

$$f(t):=\begin{cases}\operatorname{Im}(PG^n)(t), & t\in(-a_n, a_n)\\ 0, & |t|>a_n\end{cases}$$

The latter belongs to $L_p(\mathbb{R})$ any $p>0$, and since the Hilbert transform is a bounded operator in $L_p(\mathbb{R})$, for $p>1$, we conclude (recall (2.8)) that

$$\int_{\mathbb{R}}|PG^n|^p\,dx\leqslant C\int_{\mathbb{R}}|PW|^p\,dx. \tag{2.19}$$

Thus (see (2.17)), we have proved that

$$\int_{J_n}|(PW)'(x)\,\varepsilon_n(x)|^p\,dx\leqslant C\int_{\mathbb{R}}|PW|^p\,dx, \qquad p>1. \tag{2.20}$$

We proceed to prove (2.19) for the exceptional case $p=1$: As the Hilbert transform is not bounded from $L_1(\mathbb{R})$ to $L_1(\mathbb{R})$, we proceed a little differently. Let

$$\psi(z):=\frac{z}{a_n}-\left\{\left(\frac{z}{a_n}\right)^2-1\right\}^{1/2}$$

denote the conformal map of $\bar{\mathbb{C}}\setminus[-a_n, a_n]$ onto $\{w\colon |w|<1\}$. For a given $P$, we introduce the Blaschke product

$$B(z):=\prod\frac{\psi(z)-\psi(\alpha_j)}{1-\psi(z)\overline{\psi(\alpha_j)}},$$

taken over all zeros $\alpha_j$ of $P$ (according to multiplicity) in $\mathbb{C}\setminus[-a_n, a_n]$. (We take the product to be 1 if $P$ does not vanish in $\mathbb{C}\setminus[-a_n, a_n]$.)

Then $PG^n/B$ does not vanish in $\mathbb{C}\backslash[-a_n, a_n]$, so we may consider a single-valued branch

$$g(z) := (PG^n/B)(z)^{1/2}, \qquad z \in \mathbb{C}\backslash[-a_n, a_n].$$

Since $|B| = 1$ on $[-a_n, a_n]$ and $|B| < 1$ in $\mathbb{C}\backslash[-a_n, a_n]$, we obtain

$$\int |PG^n|\, d\sigma_n \leqslant \int |g|^2\, d\sigma_n.$$

Assuming $P \in \mathscr{P}_{n-2}$, we see that $g(z) = O(1/z)$ as $z \to \infty$, so that $g \in H^2$. As before, we see that the restriction of $g$ to $(-\infty, -a_n) \cup (a_n, \infty)$ is the Hilbert transform of the function

$$f_1(t) := \begin{cases} \operatorname{Im} g(t+i0), & t \in (-a_n, a_n) \\ 0, & |t| > a_n. \end{cases}$$

Then Carleson's theorem, followed by the boundedness of the Hilbert transform in $L^2(\mathbb{R})$, give

$$\begin{aligned} \int |g|^2\, d\sigma_n &\leqslant C_1 \int_{-\infty}^{\infty} |g(t+i0)|^2\, dt \\ &\leqslant C_2 \int_{-a_n}^{a_n} |g(t)|^2\, dt \\ &\leqslant C_3 \int_{-a_n}^{a_n} |PW|(t)\, dt. \end{aligned}$$

Again, we have (2.19) and hence (2.20).

*Step* 5.

We estimate the tail of the integral.

More precisely, we estimate $\|(PW)'\, \varepsilon_n\|_{L_p(\mathbb{R}\backslash J_n)}$. With $W$ replaced by $G^n$, this is easy. By (2.18), (2.8), we have for $x \notin J_n$,

$$|(PG^n)'\,(x)| \leqslant \frac{1}{\pi} \int_{-a_n}^{a_n} \frac{|(PW)(t)|}{(t-x)^2}\, dt.$$

Therefore, Hölder's inequality, and then integration with respect to $x$, yields with $q = p/(p-1)$,

$$\int_{\mathbb{R}\backslash J_n} |(PG^n)'|^p\, dx \leqslant \frac{1}{\pi^p} \left( \int_{-a_n}^{a_n} |PW|^p\, dt \right) \left\{ \int_{\mathbb{R}\backslash J_n} \left( \int_{-a_n}^{a_n} \frac{dt}{(x-t)^{2q}} \right)^{p/q} dx \right\}.$$

Since $|x \pm a_n| \geqslant a_n n^{-2/3}$ for $x \in \mathbb{R}\backslash J_n$, a simple calculation of the double integral in $\{\}$ yields $O((a_n n^{-2/3})^{-p})$, provided $p > 1$. For $p = 1$, trivial modifications are required, giving the same answer. But $\varepsilon_n(x) = a_n n^{-2/3}$ for $x \in \mathbb{R} \backslash J_n$, so that we obtain

$$\int_{\mathbb{R} \backslash J_n} |(PG^n)' \varepsilon_n|^p \, dx \leqslant C \int_{\mathbb{R}} |PW|^p \, dx. \tag{2.21}$$

Finally, write

$$(PW)' = ([PG^n][W/G^n])' = [PG^n]' \, [W/G^n] + [PG^n][W/G^n]'.$$

In Section 6, we prove that

$$|[W/G^n]'(x)| \, \varepsilon_n(x) \leqslant C, \qquad x \in \mathbb{R} \backslash J_n. \tag{2.22}$$

Now, (2.19), (2.21), (2.22) (and (2.8), (2.9)) yield

$$\int_{\mathbb{R} \backslash J_n} |(PW)' \varepsilon_n|^p \, dx \leqslant C \int_{\mathbb{R}} |PW|^p \, dx,$$

and (recalling (2.20)) the proof of Theorem 1.1 is completed.

*Remark.* In the last step, we assumed that $P \in \mathscr{P}_{n-1}$. Thus, we should have above $\varepsilon_{n-1}(x)$ instead of $\varepsilon_n(x)$. However $a_n/a_{n-1} = 1 + O(1/n)$ for $n \geqslant 1$ (see, e.g., Lemma 5.2 in [10, p. 478]) and therefore

$$\varepsilon_n(x) \sim \varepsilon_{n-1}(x)$$

uniformly for $x \in \mathbb{R}$ and $n \geqslant 1$.

## 3. Proof of Lemma 2.4

We first note that (2.3) implies in particular that

$$|\varepsilon_n'(x)| = \frac{1}{2n}\left(1 - \frac{|x|}{a_n} + n^{-2/3}\right)^{-3/2} < \frac{1}{2}$$

for $0 < |x| < a_n$. Therefore,

$$|\varepsilon_n(x) - \varepsilon_n(y)| < \tfrac{1}{2}|x - y|, \qquad x, y \in \mathbb{R}. \tag{3.1}$$

Now, fix a square $K_{x_0, h}$ of the form (2.15). A necessary condition for the semicircle

$$\Gamma_x := \{z = x + \varepsilon_n(x) \, e^{i\theta} : \theta \in [0, \pi]\}$$

to intersect $K_{x_0, h}$ is

$$|x - x_0| \leqslant \tfrac{1}{2}h + \varepsilon_n(x).$$

This implies, by (3.1), that

$$|x-x_0| \leqslant \tfrac{1}{2}h+\varepsilon_n(x_0)+\tfrac{1}{2}|x-x_0|,$$

that is

$$|x-x_0| \leqslant h+2\varepsilon_n(x_0). \tag{3.2}$$

Next, $\Gamma_x \cap K_{x_0,h}$ consists of at most two arcs, and as each such arc is convex, $\Gamma_x \cap K_{x_0,h}$ has length at most $4h$ (of course, this is a very crude estimate). Therefore, the total angular measure of $\Gamma_x \cap K_{x_0,h}$ is at most $4h/\varepsilon_n(x)$. Obviously, it does not exceed $\pi$ as well. Taking into account (3.2), we obtain, by the definition (2.12) of $\sigma_n$, that

$$\sigma_n(K_{x_0,h}) \leqslant \int_{|x-x_0| \leqslant h+2\varepsilon_n(x_0)} \min\{\pi, 4h/\varepsilon_n(x)\}\, dx. \tag{3.3}$$

We distinguish two cases:

*Case* I: $h \geqslant \varepsilon_n(x_0)$. Then the integral in (3.3) is taken over an interval of length $\leqslant 6h$, so that

$$\sigma_n(K_{x_0,h}) \leqslant 6\pi h.$$

*Case* II: $h<\varepsilon_n(x_0)$. Assume first, that

$$0 \leqslant x_0 \leqslant a_n(1-3n^{-2/3}). \tag{3.4}$$

Then a straightforward calculation (recall (2.3)) yields

$$\begin{aligned}\int_{|x-x_0| \leqslant h+2\varepsilon_n(x_0)} 4h/\varepsilon_n(x)\, dx &\leqslant \int_{x_0-3\varepsilon_n(x_0)}^{x_0+3\varepsilon_n(x_0)} (4h/\varepsilon_n(x))\, dx \\ &= 4h \cdot \frac{2}{3} R\left\{\left(1+\frac{3}{R}\right)^{3/2}-\left(1-\frac{3}{R}\right)^{3/2}\right\},\end{aligned} \tag{3.5}$$

where

$$R := n\left(1-\frac{x_0}{a_n}+n^{-2/3}\right)^{3/2} \geqslant 8,$$

by (3.4). Thus, the integral (3.5) is $\leqslant Ch$, for some absolute constant $C$.

If $x_0>a_n(1-3n^{-2/3})$, then the integral in (3.5) is taken over an interval of length $6\varepsilon_n(x_0) \leqslant 6\, a_n n^{-2/3}$, while $\varepsilon_n(x) \geqslant C\, a_n n^{-2/3}$ in this interval (see (3.1), and recall the definition of $\varepsilon_n(x)$). Thus, we again obtain the bound $Ch$ for the integral (3.5). The case $x_0 \leqslant 0$ is treated similarly.

## 4. Proof of Lemma 2.1

The proof of Lemma 2.1 is contained in our paper [9]. However, it is spread over several places there, and is carried out for the general case $A > 0$, which does not make for easy reading. Therefore we present the proof (albeit a sketchy one) for the case $A > 1$.

First, we collect some properties (cf. [9, Lemma 3.1]) that follow easily from (1.3), with $A > 1$:

$$Q'(x) \leqslant Q'(1)\, x^{A-1}, \qquad x \in (0, 1]. \tag{4.1}$$

Note that this implies that $Q$ is differentiable at 0 and $Q'(0) = 0$.

$$Q'(x) \uparrow \infty \qquad \text{as} \quad x \to \infty. \tag{4.2}$$

$$a_n x Q'(a_n x) \sim Q(a_n x) \sim n, \tag{4.3}$$

uniformly for $x \in [a, b]$, any fixed $a, b > 0$. We deduce from (4.2), (4.3) that

$$a_n/n = o(1), \qquad n \to \infty, \tag{4.4}$$

It is also shown in [9, Lemma 3.1] that

$$t^A \leqslant (xtQ'(xt))/(xQ'(x)) \leqslant t^B, \qquad x \in (0, \infty), \quad t \in (1, \infty), \tag{4.5}$$

and

$$A \leqslant xQ'(x)/Q(x) \leqslant B, \qquad x \in (0, \infty). \tag{4.6}$$

Now, let

$$z := x + \varepsilon e^{i\theta}, \qquad \varepsilon := \varepsilon_n(x).$$

By the definition (2.1) of $H_x$, we obtain

$$\begin{aligned} |H_x(z)/W(|z|)| &= \exp(Q(|z|) - Q(x) - Q'(x)(\operatorname{Re} z - x)) \\ &= \exp(Q(|z|) - Q(x) - Q'(x)\, \varepsilon \cos\theta) =: e^{\gamma}. \end{aligned} \tag{4.7}$$

To prove (2.4), it suffices to show that $\gamma = O(1)$, uniformly for $x \in J_n$ and $z$ of the above form.

*Case* I: $0 \leqslant x \leqslant 4a_n/n$. Then by (4.4), $x = o(1)$. Also,

$$\varepsilon_n(x) = O(a_n/n) = o(1),$$

in the range considered. Thus $\gamma$ in (4.7) is $o(1)$.

*Case* II. $4a_n/n \leqslant x \leqslant a_n/2$. Then

$$x/\varepsilon = n\frac{x}{a_n}\left(1-\frac{x}{a_n}+n^{-2/3}\right)^{1/2} > 2.$$

By Lemma 2.1 in [9, p. 1069],

$$\gamma \leqslant C(2x)\, Q'(2x)(\varepsilon/(x-\varepsilon))^2 \leqslant C_1\, Q'(2x)\, \varepsilon^2/x. \tag{4.8}$$

Applying (4.2), (4.3), we obtain that

$$\gamma \leqslant C_2 \frac{n}{a_n} 4 \frac{n}{a_n} \varepsilon^2.$$

Since

$$\varepsilon = O(a_n/n)$$

in the range considered, we obtain

$$\gamma = O(1).$$

*Case* III: $a_n/2 \leqslant x \leqslant 2a_n$. Since $\varepsilon_n(x) \leqslant a_n n^{-2/3}$, $x \in \mathbb{R}$, we see that $x/\varepsilon > 2$ for the present range as well. Then (4.8) yields

$$\gamma \leqslant C \frac{n}{a_n}\frac{1}{a_n} a_n^2 n^{-4/3} = Cn^{-1/3}.$$

## 5. Proof of Lemma 2.2

We begin with

LEMMA 5.1. *Assume the conditions of Theorem 1.2. Define for* $x \in [-1, 1]\backslash\{0\}$, $n \geqslant 1$,

$$\mu_n(x) := \frac{2}{\pi}\int_0^1 \frac{(1-x^2)^{1/2}}{(1-t^2)^{1/2}} \frac{a_n t Q'(a_n t) - a_n x Q'(a_n x)}{n(t^2-x^2)}\, dt, \tag{5.1}$$

*Then* $\mu_n(x) > 0$ *for* $x \in (-1, 1)\backslash\{0\}$ *and*

$$\int_{-1}^{1} \mu_n(x)\, dx = 1. \tag{5.2}$$

*Next, for* $z \in \mathbb{C}$, *let*

$$U_n(z) := \int_{-1}^{1} \log|z-t|\, \mu_n(t)\, dt - \frac{1}{n} Q(a_n|z|) + \frac{1}{n}\chi_n, \tag{5.3}$$

*where*

$$\chi_n := \frac{2}{\pi} \int_0^1 \frac{Q(a_n t)}{(1-t^2)^{1/2}}\, dt + n \log 2.$$

*Then $U_n$ is an even continuous function in $\mathbb{C}$ and satisfies*

$$U_n(x) = 0, \qquad x \in [-1, 1]; \tag{5.4}$$

$$U_n(x) < 0; \qquad U_n'(x) < 0, \qquad x \in (1, \infty). \tag{5.5}$$

*Furthermore for some $C_1$, $C_2$, $\delta_0$,*

$$-C_1\, \delta^{3/2} \leqslant U_n(1+\delta) \leqslant -C_2\, \delta^{3/2}, \qquad \delta \in [0, \delta_0], \tag{5.6}$$

*and given $K > 0$, there exists $C_3 = C_3(K)$ such that*

$$U_n(x) \leqslant -C_3 \log x, \qquad x \geqslant 1 + K. \tag{5.7}$$

*Proof.* See [12, pp. 37–39, 45, 55]. ∎

Now we can give an explicit expression for the function $G$ discussed in Lemma 2.2. Set

$$G(z) := \exp\left( -\int_{-1}^{1} \log(z/a_n - t)\, \mu_n(t)\, dt - \frac{1}{n} \chi_n \right), \tag{5.8}$$

where log denotes the principal branch. Note that (5.2) ensures that $G$ is single-valued in $\mathbb{C} \setminus [-a_n, a_n]$ and that it has a simple zero at infinity. Since $\mu_n(t)$ is real-valued, we also obtain $G(\bar{z}) = \overline{G(z)}$. Next, by (5.3), (5.8), we have

$$W(|z|) = e^{-Q(|z|)} = e^{nU_n(z/a_n)} |G^n(z)|. \tag{5.9}$$

Therefore, (2.8), (2.9) follow by (5.4), (5.5), so we have completed the proof of part (a) of Lemma 2.2. We turn to the proof of part (b). In view of (5.9), we need to show that $nU_n(z/a_n)$ is bounded from above, for the relevant range of $z$.

LEMMA 5.2. *Assume the conditions of Theorem 1.1, and let $0 < \eta < 1$. Then for $t \in [0, 1]$ and for $n$ large enough, there holds:*

$$U_n(s+it) \leqslant C_1 t, \qquad s \in [0, \eta]; \tag{5.10}$$

$$U_n(s+it) C_2 \max\{t^{3/2}, t(1-s)^{1/2}\}, \qquad s \in [\eta, 1]; \tag{5.11}$$

$$U_n(s+it) \leqslant C_3(t^{3/2} - C_4(s-1)^{3/2}), \qquad s \in [1, 2]. \tag{5.12}$$

*Furthermore,* (5.11), (5.12) *hold if* $W$ *only satisfies the conditions of Theorem* 1.2.

*Proof.* The above estimates are contained in [9, Lemma 4.2, 4.3, 4.4]. For the reader's convience, we prove (5.10), since this inequality was stated in [9, Lemma 4.2] in a different form. By (5.3), (5.4),

$$\begin{aligned} U_n(s+it) &= U_n(s+it) - U_n(s) \\ &= \frac{1}{2}\int_{-1}^{1} \log\left(1+\left(\frac{t}{s-u}\right)^2\right) \mu_n(u)\,du \\ &\quad + \left\{\frac{Q(a_n|s|) - Q(a_n(s^2+t^2)^{1/2})}{n}\right\} \\ &\leqslant \int_0^1 \log\left(1+\left(\frac{t}{s-u}\right)^2\right) \mu_n(u)\,du, \end{aligned} \tag{5.13}$$

by monotonicity of $Q$ and evenness of $\mu_n$. Next, by Lemma 4.1 in [9],

$$\mu_n(x) \leqslant C\sqrt{1-x^2}, \qquad x \in [\eta, 1], \tag{5.14}$$

and

$$\mu_n(x) \leqslant C\,\psi_n(x), \qquad x \in (0, \eta], \tag{5.15}$$

where

$$\psi_n(x) := \int_x^2 \frac{a_n Q'(a_n t)}{nt}\,dt. \tag{5.16}$$

The substitution $u = a_n t$ yields

$$\psi_n(x) = \frac{a_n}{n}\int_{a_n x}^{2a_n} \frac{Q'(u)}{u}\,du = O(1),$$

by (4.1) and (4.5). Thus,

$$\mu_n(x) \leqslant C\sqrt{1-x^2}, \qquad x \in [0, 1], \tag{5.17}$$

and we deduce from (5.13) that

$$U_n(s+it) \leqslant C_1 \int_0^1 \log\left(1+\left(\frac{t}{s-u}\right)^2\right) du.$$

The substitutions $s-u=ty$ gives

$$U_n(s+it) \leqslant C_1 t \int_{-\infty}^{\infty} \log(1+y^{-2})\,dy \leqslant C_2 t. \qquad \blacksquare$$

Now we can prove the inequality (2.10) in part (b) of Lemma 2.2. By (5.9), it is equivalent to

$$nU_n(s+it) \leqslant C, \tag{5.18}$$

for all $s$, $t$ of the form

$$\begin{cases} s = \dfrac{1}{a_n}(x + \varepsilon_n(x)\cos\theta) \\ t = \dfrac{1}{a_n}\varepsilon_n(x)\sin\theta \end{cases}, \qquad x \in J_n,\ \theta \in [0, \pi]. \tag{5.19}$$

*Case* I. $0 \leqslant x \leqslant \frac{1}{2}a_n$. Then $\varepsilon_n(x) \leqslant C\, a_n/n$, so that $0 \leqslant s \leqslant 1/2 + C/n$, $0 \leqslant t \leqslant C/n$. Applying (for $n$ large enough) (5.10), we obtain (5.18).

*Case* II. $\frac{1}{2}a_n \leqslant x \leqslant a_n$. Then

$$t = \frac{1}{a_n}\varepsilon_n(x)\sin\theta \leqslant n^{-2/3},$$

and since

$$\frac{1}{a_n}\varepsilon_n(x) = \frac{1}{n}\left(1 - \frac{x}{a_n} + n^{-2/3}\right)^{-1/2} \leqslant \frac{1}{n}(1 - s - n^{-2/3} + n^{-2/3})^{-1/2}$$

(see the definition (5.19) of $s$), we obtain

$$t \leqslant \frac{1}{n}(1-s)^{-1/2}.$$

Applying (5.11), we again obtain (5.18).

*Case* III. $a_n \leqslant x \leqslant a_n(1 + n^{-2/3})$. In this case, we apply (5.12) and get (5.18)

This proves (5.18) for $x > 0$. Since $U(\bar{z}) = U(z)$ and $U$ is even, the proof of (5.18) is complete. Note that (5.11), (5.12) hold also if $W$ only satisfies the conditions of Theorem 1.2 (see the last assertion of Lemma 5.2). This concludes the proof of Lemma 2.2.

## 6. Proof of (2.22)

For $x \geqslant a_n(1 + n^{-2/3})$, we deduce from (5.9), (2.3) that

$$(W/G^n)'(x)\,\varepsilon_n(x) = n^{1/3}\,U_n'(x/a_n)\exp(nU_n(x/a_n)) =: \Delta. \tag{6.1}$$

We consider three ranges of $x$:

*Case* I. $a_n(1+n^{-2/3}) \leqslant x \leqslant a_n(1+\delta)$, some small enough $\delta > 0$. Then by (5.6),

$$U_n(x/a_n) \leqslant -C_1(x/a_n - 1)^{3/2}. \tag{6.2}$$

Also (cf. [12, pp. 39, 55]),

$$0 \geqslant U_n'(x/a_n) \geqslant -C_2(x/a_n - 1)^{1/2}, \tag{6.3}$$

for the range considered. Therefore,

$$\begin{aligned} |\Delta| &\leqslant C_3\, n^{1/3}(x/a_n - 1)^{1/2} \exp(-C_1\, n(x/a_n - 1)^{3/2}) \\ &= C_3\, R^{1/2} \exp(-C_1\, R^{3/2}), \end{aligned}$$

where

$$R := n^{2/3}(x/a_n - 1) \geqslant 1.$$

So,

$$|\Delta| \leqslant C_4.$$

*Case* II: $a_n(1+\delta) \leqslant x \leqslant Ka_n$, some $K > 0$ large enough. Here

$$U_n(x/a_n) \leqslant U_n(1+\delta) \leqslant -C_5,$$

by (5.5), and

$$U_n'(x/a_n) = \int_{-1}^{1} (x/a_n - t)^{-1}\, \mu_n(t)\, dt - a_n\, Q'(x)/n. \tag{6.4}$$

Since

$$a_n\, Q'(x)/n \sim \frac{a_n}{x} \sim 1$$

for the range considered (see (4.3)), we see that

$$U_n'(x/a_n) = O(1).$$

Thus again

$$|\Delta| \leqslant C_6\, n^{1/3} \exp(-n\, C_5) \leqslant C_7.$$

*Case* III. $x \geqslant Ka_n$. Since $Q(a_n) \sim n$ (by (4.3)), we have

$$\frac{1}{n}\chi_n = O(1),$$

by the definition of $\chi_n$ in Lemma 5.1. Then (5.2), (5.3) imply that

$$U_n(x/a_n) \leqslant \log(x/a_n - 1) - \frac{1}{n} Q(x) + O(1).$$

Now by (4.3), (4.5), and (4.6),

$$\frac{1}{n} Q(x) \geqslant C_8 \frac{xQ'(x)}{a_n Q'(a_n)} \geqslant C_8 \left(\frac{x}{a_n}\right)^A \geqslant 2[\log(x/a_n - 1) + O(1)],$$

for $x \geqslant Ka_n$, $K$ large enough, so

$$U_n(x/a_n) \leqslant -\frac{1}{2n} Q(x).$$

Also, by (6.4),

$$|U_n'(x/a_n)| \leqslant C_9 + a_n Q'(x)/n \leqslant C_9 + C_{10}\, a_n\, Q(x)/(nx) \leqslant C_{11} Q(x)/n,$$

by (4.6). Therefore

$$\begin{aligned} |\Delta| &\leqslant C_{12} n^{1/3} \cdot (Q(x)/n) \cdot e^{-Q(x)/2} \\ &\leqslant C_{12} n^{-2/3} \cdot Q(x) \cdot e^{-Q(x)} \leqslant C_{13}. \end{aligned}$$

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